

A Note On the Use of Fiducial Limits for Control Charts

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Abstract

Many of the early works in the quality control literature construct control limits through the use of graphs and tables as described in Wortham and Ringer (1972). However, the methods used in this literature are restricted to using only the values that the graphs and tables can provide and to the case where the parameters of the underlying distribution are known. In this note, we briefly describe a technique which can be used to calculate exact control limits without the use of graphs or tables. We also describe what are commonly referred to in the literature as fiducial limits. Fiducial limits are often used as the limits in control charting when the parameters of the underlying distribution are unknown.

1 Introduction

In many applications using attribute (count) data, the number of nonconformities often has discrete distributions such as the binomial, Poisson and geometric. In this note, we provide the fiducial limits for these distributions because, in certain cases, these limits are often used as the upper and lower limits for control charts.

Early works related to obtaining exact control limits were based on the use of graphs and tables. For example, Wortham and Ringer (1972) provide a good review on exact control limits using graphs and tables when the true parameter of the underlying distribution is known. These graphs and tables were used to approximate or interpolate the values needed to obtain an interval with $1 - \alpha$ confidence level when the parameters of the underlying distribution were known. Before the advent of powerful computers and software, it was often troublesome and time-consuming to compute exact control limits. Thus, the methods that were based on the use of tables and graphs were prevalent in the literature because of their simplicity and convenience.

Unfortunately, the studies that use these graphs and tables are restricted to using the specific confidence interval values that can be inferred from these tables and graphs. Additionally, errors are difficult to avoid because of the required use of visual interpolation. However, with powerful computers and accessible software available, it has now become quite trivial to calculate exact control limits without the use of graphs and tables. Exact control limits are obtained by calculating the corresponding quantiles of the underlying distribution involved. These days, there are many statistical software programs that provide the quantiles for any reasonably common distribution. For example, in R Development Core Team (2008), a non-commercial open source statistical software package for statistical computing and graphics, the binomial and Poisson quantile functions are given by `qbinom()` and `qpois()` respectively. R can be obtained at no cost from <http://www.r-project.org>. Other commercial statistical software such as Minitab and SAS also provide the quantile functions for the commonly known distributions.

Another shortcoming of the control limit methods that use the graphs and tables in Wortham and Ringer (1972) is that, in order to use these tables, one has to assume that the parameter of the underlying distribution is known. In many practical examples, the true parameters are not known. As aforementioned, if the true parameter is known, the exact limits are easily calculated by using the quantile functions in the various statistical software packages available. If the parameters are unknown, then the control limits can be obtained by inverting the relation between the tails and the parameters. These control limits are then referred to as fiducial limits in the statistics literature and are explained in more detail in the following section.

2 Fiducial Limits for the discrete distributions

Many authors, including Clopper and Pearson (1934), Garwood (1936), Stevens (1950), Bickel and Doksum (1977), Kendall and Stuart (1979), mention the exact or fiducial confidence limits for several discrete distributions, along with the classical confidence limits based on the central limit theorem. In this note, we provide a review of several fiducial limits which can be useful for various control charting methods including the p , u , c and g charts.

2.1 The binomial distribution

Let X denote a binomial random variable with size n and Bernoulli probability p . The fiducial limits for p are obtained by inverting the following two equal tails for p

$$\sum_{i=0}^x \binom{n}{i} p^i (1-p)^{n-i} = \frac{\alpha}{2} \quad (1)$$

and

$$\sum_{i=x}^n \binom{n}{i} p^i (1-p)^{n-i} = \frac{\alpha}{2}, \quad (2)$$

where $x = 1, 2, \dots, n - 1$. The lower and upper limits with $1 - \alpha$ confidence level are given by

$$p_L = \left\{ 1 + \frac{n - x + 1}{x \cdot F_{2x, 2(n-x+1)}^{(\alpha/2)}} \right\}^{-1} \quad (3)$$

and

$$p_U = \left\{ 1 + \frac{n - x}{(x + 1) \cdot F_{2(x+1), 2(n-x)}^{(1-\alpha/2)}} \right\}^{-1}, \quad (4)$$

where $F_{\nu_1, \nu_2}^{(\xi)}$ denotes the ξ quantile of the F distribution with degrees of freedom ν_1 and ν_2 . Here, the lower limit is 0 if $x = 0$ and the upper limit is 1 if $x = n$. Derivations of the fiducial limits for p based on the F distribution are given by Blyth (1986), Hald (1952), and Leemis and Trivedi (1996). It should be noted that the quantile of the F distribution is calculated by using `qf()` in R.

2.2 The Poisson distribution

Garwood (1936) provide the fiducial limits for the Poisson distribution. Let X_1, X_2, \dots, X_n be a random sample from a Poisson distribution with mean λ . For convenience, let $Y = \sum_{j=1}^n X_j$. Then, the random variable Y has a Poisson distribution with mean $n\lambda$. Let $Y = y$, then the lower and upper limits with $1 - \alpha$ confidence level are given by

$$\lambda_L = \frac{1}{2n} \Gamma_{y, 2}^{(\alpha/2)} \quad (5)$$

and

$$\lambda_U = \frac{1}{2n} \Gamma_{y+1, 2}^{(1-\alpha/2)}, \quad (6)$$

where $\Gamma_{a, b}^{(\xi)}$ is the ξ quantile of the gamma distribution with parameters a and b . Here the lower limit is 0 if $y = 0$. It should be noted that the quantile of the gamma distribution is calculated by using `qgamma()` in R.

2.3 The geometric distribution

Let X_1, X_2, \dots, X_n be a random sample from a geometric distribution with Bernoulli parameter p . For convenience, let $Y = \sum_{j=1}^n X_j$. Then, the random variable Y has a negative binomial distribution with size n and Bernoulli parameter p . Let $Y = y$, then the lower and upper limits with $1 - \alpha$ confidence level equal

$$p_L = \left\{ 1 + \frac{y+1}{n} \cdot F_{2(y+1), 2n}^{(1-\alpha/2)} \right\}^{-1} \quad (7)$$

and

$$p_U = \left\{ \frac{n \cdot F_{2n, 2y}^{(1-\alpha/2)}}{y + n \cdot F_{2n, 2y}^{(1-\alpha/2)}} \right\}^{-1}, \quad (8)$$

where the upper limit is 1 if $y = 0$. These limits are provided in Exercise 9.22 of Casella and Berger (2002).

3 Illustrative Examples

References

- Bickel, P. J. and Doksum, K. A. (1977). *Mathematical statistics : basic ideas and selected topics*. Holden-Day, San Francisco.
- Blyth, C. R. (1986). Approximate binomial confidence limits. *Journal of the American Statistical Association*, 81:843–855.
- Casella, G. and Berger, R. L. (2002). *Statistical Inference*. Duxbury, second edition.
- Clopper, C. J. and Pearson, E. S. (1934). The use of confidence or fiducial limits illustrated in the case of the binomial. *Biometrika*, 26:404–413.
- Garwood, F. (1936). Fiducial limits for the Poisson distribution. *Biometrika*, 28:437–442.

- Hald, A. (1952). *Statistical Theory with Engineering Applications*. John Wiley, New York.
- Kendall, M. and Stuart, A. (1979). *The Advanced Theory of Statistics*, volume 2. Charles Griffin, fourth edition.
- Leemis, L. M. and Trivedi, K. S. (1996). A comparison of approximate interval estimators for the bernoulli parameter. *The American Statistician*, 50:63–68.
- R Development Core Team (2008). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria. ISBN 3-900051-07-0.
- Stevens, W. L. (1950). Fiducial limits of the parameter of a discontinuous distribution. *Biometrika*, 37:117–129.
- Wortham, A. W. and Ringer, L. J. (1972). A note on exact control limits. *International Journal of Production Research*, 10:193–200.